UNIT - V

Section - 1

Introduction:

We Consider The General 1^{st} order eqn

$$y' = f(x,y) \rightarrow (1)$$

Where f is some continuous function only in certain special cases it is possible to find explicit analytic expression for the soln of (1)

One such special case is the linear equation

$$y' = g(x)y + h(x) \rightarrow (2)$$

Where, g, h are continuous on some interval I.

Any soln φ of (2) can be written in the form,

$$\varphi(x) = e^{\theta x}\int_{x_0}^{x} e^{-\theta(t)}\, h(t)\, dt + c\, e^{\theta x} \rightarrow (3)$$

where $\theta(x) = \int_{x_0}^{x} g(t)\, dt$, x_0 in I and c is constant.

Our main purpose it P.T a wide class of eqn of the form (1) have soln and that soln of initial value problem are unique.

If f is not a linear eqn there are certain limitations which must be expected. Concerning any general existence Theorem.

For example:

Consider The eqn

$$y' = y^2$$

Here $f(x,y) = y^2$

We see that f has derivatives of all orders with respect to x and y at every point in the (x,y) plane.

A soln φ of this eqn satisfying the initial condition $\varphi(1) = -1$ is given by $\varphi(x) = \frac{-1}{x}$

This soln cases to exist at $x=0$.

This eg shows that any general existence at Theorem for ① can only assert the existence of a soln on some interval near by the initial point.

This does not occur in the case of the linear eqn ③ for it is clear from ③ that only soln φ exists on all of the interval I.

Section-2

Equation with Variable Seperable

Theorem:

Let, g, h be continuous real valued function for $a \le x \le b$, $c \le y \le d$ respectively and consider the eqn.

$$h(y)\, y' = g(x) \rightarrow (*)$$

Here G, H are any functions such that $G' = g$, $H' = h$ and c is any constant such that the relation

$$H(y) = G(x) + c$$

Define a real valued differentiable function φ for x in some interval I contained in $a \le x \le b$ that φ will be soln of $(*)$ on I.

Conversely if φ is a soln of $(*)$ on I, it satisfies the relation $H(y) = G(x) + c$ on I for some constant c.

Proof:

A 1^{st} order eqn

$$y' = f(x, y) \rightarrow ①$$

is said to have the variables seperated if f can be written in the form.

$$f(x, y) = \frac{g(x)}{h(y)},$$

Where g, h are fun of a single argument.

In this case we write the eqn as

$$h(y)\cdot\frac{dy}{dx} = g(x) \rightarrow (2)$$

$$h(y)\,dy = g(x)\,dx$$

Let us consider the case as g and h are continuous real valued function, defined for real x and y respectively.

If $y=\varphi(x)$ is a real valued soln of (2) on some interval containing at point x_0 then,

$$h[\varphi(x)]\,\varphi'(x) = g(x) \quad \forall\ x \text{ in } I$$

$$\int_{x_0}^{x} h[\varphi(t)]\,\varphi'(t)\,dt = \int_{x_0}^{x} g(t)\,dt \rightarrow (3) \quad \forall x \text{ in } I$$

Let $u=\varphi(t)$ in the L.H.S of (3) we get,

$$\int_{\varphi(x_0)}^{\varphi(x)} h(u)\,du = \int_{x_0}^{x} g(t)\,dt$$

Conversely,

suppose x and y are related by the formula

$$\int_{y_0}^{y} h(u)\,du = \int_{x_0}^{x} g(t)\,dt \rightarrow (4)$$

and that this defines implicity a differentiable function φ for x in I

Then this function satisfies

$$\int_{y_0}^{\varphi(x)} h(u)\,du = \int_{x_0}^{x} g(t)\,dt. \quad \forall x \text{ in } I$$

Diff (2) w.r.t x we obtain,

$$h[\varphi(x)]\,\varphi'(x) = g(x)$$

Which S.T φ is a soln at (2) on I

In practise, the usual way of with (2) is to write t as

$h(y)dy = g(x)dx$ (Thus seperated the variables)

Integrate we get,

$\int h(y)dy = \int g(x)dx + c$ where c is a constant

Then integral are anti derivative.

Thus $H(y) = \int h(y)\,dy$ and $G(x) = \int g(x)dx$

Represents any two functions H and G such that

$H' = h$ and $G' = g$

Thus any diff fun φ which is defined implicity by the relation.

$H(y) = G(x) + C \rightarrow (5)$

will be soln of (2)

* We say that a relation $F(x,y) = 0$ defines a fun φ implicity for x in some interval I. If for each x in I there is a y such that $F(x,y) = 0$ this y being denoted by $\varphi(x)$

* If fun φ will be a soln of $y' = \frac{g(x)}{h(y)}$ on I provided $h[\varphi(x)] \neq 0$ $\forall$ x in I.

Problems,

1.a) Find all real valued soln of $y' = x^2 y$

Soln:

Given eqn is $y' = x^2 y$

$\Rightarrow \frac{dy}{dx} = x^2 y$

$\frac{dy}{y} = x^2 dx$

If $y \neq 0$, Integrating we get,

$\log y = \frac{x^3}{3} + \log c$

$\log y - \log c = x^3/3$

$\log\left(\frac{y}{c}\right) = x^3/3$

$y = c\, e^{x^3/3}$

b) Solve $y' = \frac{e^{x-y}}{1+e^x}$

Soln:

Given eqn is $y' = \frac{e^{x-y}}{1+e^x}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x \cdot e^{-y}}{1+e^x}$

$e^y \, dy = \frac{e^x}{1+e^x} dx$

Integrating,

$e^y = \log(1+e^x) + c$

pl. 180 (2)

$y' = y^2 \Rightarrow \frac{dy}{y^2} = dx$

$y^{-2} dy = dx \Rightarrow -\frac{1}{y} = x + c$

$x + \frac{1}{y} = c$

$(x_0, y_0) \Rightarrow \frac{x_0 y_0 + 1}{y_0} = c$

$x + \frac{1}{y} = \frac{x_0 y_0 + 1}{y_0}$

$\frac{1}{y} = \frac{x_0 y_0 + 1 - x y_0}{y_0}$

$y = \frac{y_0}{1 - y_0(x - x_0)} = \phi(x)$

1.c) (i) Find the soln of $y' = 2y^{1/2}$ passing through the point (x_0, y_0) where $y_0 > 0$

(ii) Find the all soln of this problem passing through $(x_0, 0)$

Soln:

(i) Given eqn is

$y' = 2y^{1/2}$

$\frac{dy}{dx} = 2y^{1/2}$

$y^{-1/2} dy = 2dx$

Integrating we get,

$\frac{y^{1/2}}{1/2} = 2x + c$

$2y^{1/2} = 2x + c$

$y^{1/2} = x + c \rightarrow (1)$

$\Rightarrow c = y^{1/2} - x$

This passes through (x_0, y_0)

$\Rightarrow c = \sqrt{y_0} - x_0$

Sub we get,

$\sqrt{y} = x + \sqrt{y_0} - x_0$

$\Rightarrow y = \varphi(x) = (x - x_0 + \sqrt{y_0})^2$ for $x \geq x_0 - \sqrt{y_0}$

and $\varphi(x) = -(x - x_0 + \sqrt{y_0})^2$ for $x < x_0 - \sqrt{y_0}$

(2) Show that the solution ϕ of $y' = y^2$ which passes through the point (x_0, y_0) is given by $\phi(x) = \frac{y_0}{1 - y_0(x - x_0)}$.

Soln: $y' = y^2 \Rightarrow \frac{dy}{y^2} = dx$

Integrating, we get $-\frac{1}{y} = x + c$

$\Rightarrow x + \frac{1}{y} = c \rightarrow (1)$

It passes through the point (x_0, y_0)

$\therefore \Rightarrow x_0 + \frac{1}{y_0} = c$

$(1) \Rightarrow x + \frac{1}{y} = x_0 + \frac{1}{y_0}$

$\Rightarrow \frac{1}{y} = \frac{x_0 y_0 + 1}{y_0} - x$

$\Rightarrow \frac{1}{y} = \frac{x_0 y_0 + 1 - x y_0}{y_0}$

$\Rightarrow \boxed{y = \phi(x) = \frac{y_0}{1 - y_0(x - x_0)}}$

(ii) If it passes through $(x_0, 0)$

$\Rightarrow c = y^{1/2} - x$

$c = -x_0$

$\therefore \sqrt{y} = x - x_0$

$\Rightarrow y = \phi(x) = (x - x_0)^2$

1.d) Find all real valued soln of the following eqn (i) $y' = 3y^{2/3}$

Soln:

Given eqn is $y' = 3y^{2/3}$

$$\frac{dy}{dx} = 3y^{2/3}$$

$$\frac{dy}{y^{2/3}} = 3dx \Rightarrow y^{-2/3}dy = 3dx$$

Integrating w.r.t.

$$\frac{y^{1/3}}{1/3} = 3x + c$$

$$3y^{1/3} = 3x + c$$

$$y^{1/3} = x + c$$

$\Rightarrow y = (x+c)^3$ where c is constant.

Thus the function ϕ given by

$\phi(x) = (x+c)^3$ will be a soln of given eqn for any constant c

1.e) $y' = y^2$

Soln:

Given eqn $y' = y^2$

$$\frac{dy}{dx} = y^2$$

$$y^{-2}dy = dx$$

Integrating

$$-\frac{1}{y} = x + c$$

$$y = \frac{-1}{x+c}$$

is a soln of given eqn provided $x \neq -c$

$yy' = x$

Soln:

Given eqn is $yy' = x$

$y\,dy = x\,dx$

Integrating we get,

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$\Rightarrow y^2 = x^2 + c$

$\Rightarrow y = x + c$ is a soln

Formula

$$\int \frac{dy}{y^2 - a^2} = \frac{1}{2a} \log \frac{y-a}{y+a}$$

U.Q.

1.g) $y' = \dfrac{x + x^2}{y - y^2}$

Soln:

$$\frac{dy}{dx} = \frac{x + x^2}{y - y^2}$$

$(y - y^2)\,dy = (x + x^2)\,dx$

Integrating we get,

$$\frac{y^2}{2} - \frac{y^3}{3} = \frac{x^2}{2} + \frac{x^3}{3} + c$$

$$\frac{3y^2 - 2y^3}{6} = \frac{3x^2 + 2x^3}{6} + c$$

$\Rightarrow 3y^2 - 2y^3 = 3x^2 + 2x^3 + c$

$y^2(3 - 2y) = x^2(3 + 2x) + c$

$y' = x^2y^2 - 4x^2$

$\frac{dy}{dx} = x^2y^2 - 4x^2$

$= x^2(y^2 - 4)$

$\frac{dy}{y^2 - 4} = x^2\,dx$

$\int \Rightarrow \frac{1}{4} \log \left| \frac{y-2}{y+2} \right| = \frac{x^3}{3} + c$

$\Rightarrow \log \left| \frac{y-2}{y+2} \right| = \frac{4x^3}{3} + c$

Section-3

Exact Equation

Suppose the I^{st} order equation, $y' = f(x, y)$ is written as the form,

$$y' = -\frac{M(x,y)}{N(x,y)}$$

(or) equivalently $M(x,y) + N(x,y)\,y' = 0 \rightarrow (1)$

Where M, N are real valued function defined for real x, y on some rectangle R

The eqn ① is said to be exact in R. If Ji a function F having continuous first partial derivatives there such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ in R

Theorem: 2

Suppose the equation $M(x,y) + N(x,y)\, y' = 0$ is a exact in → ① a rectangle R and F is a real valued function such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$. Every diff fun φ defined implicity by a relation $F(x,y) = c$, c is a constant is a soln of ① and every soln of ① whose graph lies in R arives this way.

Soln:

Suppose the I^{st} order equation, $y' = f(x,y)$ is written in the form.

$$y' = -\frac{M(x,y)}{N(x,y)}$$

(or) $M(x,y) + N(x,y)\, y' = 0 \rightarrow ①$

where, M, N are real valued function defined for real x, y on some rectangle R.

① is said to be exact in R if their exist a fun F having continuous I^{st} partial derivatives in R such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \text{ in } R \rightarrow ②$$

If ① is exact in R and F is a function satisfying ② then ① becomes

$$\frac{\partial F(x,y)}{\partial x} + \frac{\partial F(x,y)}{\partial y}\, y' = 0$$

If φ is any soln on some interval I, we have

$$\frac{\partial F(x,\varphi(x))}{\partial x} + \frac{\partial F(x,\varphi(x))}{\partial y}\, y' = 0$$

If φ is any soln on some interval I, we have.

$$\frac{\partial F(x,\varphi(x))}{\partial x} + \frac{\partial F(x,\varphi(x))}{\partial y}\varphi'(x) = 0 \rightarrow (3) \quad \forall\ x \text{ in } I$$

If $\Phi(x) = F(x,\varphi(x))$ Then $\quad \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} = 0.$

(3) $\Rightarrow$ $\Phi'(x) = 0$ and hence.

$$F(x,\varphi(x)) = c \rightarrow (4) \quad \text{Where } c \text{ is some Constant}$$

i.e) $\Phi(x) = c$

Thus The soln φ must be a function which is given implicitly by The relation.

$$F(x,y) = c \rightarrow (5)$$

Conversely.

we see That if φ is a diff function on some interval I defined implicity by The relation (4) Then

$$F(x,\varphi(x)) = c \quad \forall\ x \text{ in } I$$

Diff (3) we get, Thus φ is a soln of (1)

Remark:

How to recognize when an equation is exact.

Let us suppose That

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ is exact and}$$

F is a function which has continuous second order derivative Such That.

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

Then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M}{\partial y}$, $\quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$

Since for such a function $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$

We must have,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is true That if This eqn is Valid Then The equation is exact.

Theorem: 3 Necessary and sufficient condition for an equation to be exact

Let M, N be the two real valued function which have continuous 1^{st} order partial derivatives on some rectangle,

$R: |x-x_0| \le a, \quad |y-y_0| \le b$, Then the eqn $M(x,y)+N(x,y)y'=0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R

Proof:

Assume that $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact and F is a function which has continuous second order derivatives such that,

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

Then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$

Since for such that a function

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow (1)$$

conversely,

Suppose (1) is satisfied in R. we need to find a function F satisfying

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

If we have such a function Then.

$$F(x,y) - F(x_0,y_0) = F(x,y) - F(x_0,y) + F(x_0,y) - F(x_0,y_0)$$

$$= \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y)\,ds + \int_{y_0}^{y} \frac{\partial F}{\partial y}(x_0,t)\,dt$$

$$= \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \longrightarrow (*)$$

Similarly, we could have

$$F(x,y) - F(x_0,y_0) = F(x,y) - F(x,y_0) + F(x,y_0) - F(x_0,y_0)$$

$$= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y_0)\,ds \rightarrow ⑤$$

$$= \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds \rightarrow (\ast F)$$

We now define F by the formula

$$F(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \rightarrow ③$$

This definition implies that $F(x_0,y_0)=0$ and that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \quad \forall\ x,y \text{ in } R$$

from ② it is clear that F is also given by

$$F(x,y) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds \rightarrow ④$$

This is infact true and is a consequence of the assumption ①

from ④, we have,

$$\frac{\partial F}{\partial y}(x,y) = N(x,y), \quad \forall\ x,y \text{ in } R$$

In order to show that ④ is valid.

Where F is a function given by ③. let us consider the difference

$$F(x,y) - \left[\int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds\right]$$

use ③

$$= \int_{x_0}^{x} [M(s,y) - M(s,y_0)]\,ds - \int_{y_0}^{y} [N(x,t) - N(x_0,t)]\,dt$$

$$= \int_{x_0}^{x}\int_{y_0}^{y} \frac{\partial M}{\partial y}(s,t)\,dt\,ds - \int_{y_0}^{y}\int_{x_0}^{x} \frac{\partial N}{\partial x}(s,t)\,ds\,dt$$

$$= \int_{x_0}^{x}\int_{y_0}^{y} \left[\frac{\partial M}{\partial y}(s,t) - \frac{\partial N}{\partial x}(s,t)\right] ds\,dt$$

$= 0$ by ①

$$F(x,y) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds$$

This completes the proof

To Solve the eqn $Mdx + Ndy = 0$

(i) Integrate M w.r.t x keeping y constant

(ii) Integrate those terms in N not containing x w.r.t y

(iii) The sum of those two integrals equate to the soln

2.a) Solve the eqn $2xy\,dx + (x^2+3y^2)dy = 0$. after checking the exactness

Soln:

Given eqn is $2xy\,dx + (x^2+3y^2)dy = 0$

$M = 2xy$, $N = x^2+3y^2$

$\Rightarrow \frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ This eqn is exact for all (x,y)

W.K.T There is a F such that

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = 2xy$

for each fixed y, $F = x^2y + f(y)$, where f is independent of x

$\frac{\partial F}{\partial y} = N$ gives $x^2 + f'(y) = x^2+3y^2$

$f'(y) = 3y^2$

$f(y) = y^3$

$F(x,y) = x^2y + y^3$

This the soln is

$c = x^2y + y^3 + c$, where c is constant.

2.b) Solve $[2ye^{2x} + 2x\cos y]dx + (e^{2x} - x^2\sin y)dy = 0$

Soln:

$M = 2ye^{2x} + 2x\cos y$, $N = e^{2x} - x^2\sin y$

$\frac{\partial M}{\partial y} = 2e^{2x} - 2x\sin y$, $\frac{\partial N}{\partial x} = 2e^{2x} - 2x\sin y$

This eqn is exact.

$\therefore$ There is a F such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$\Rightarrow \frac{\partial F}{\partial x} = 2ye^{2x} + 2x\cos y$

for fixed y, we get, $F = ye^{2x} + x^2\cos y + f(y)$

Where f is independent of x.

Now,

$$\frac{\partial F}{\partial y} = N$$

$$e^{2x} - x^2\sin y + f'(y) = e^{2x} - x^2\sin y$$

$$\Rightarrow f'(y) = 0 \Rightarrow f(y) = \text{constant } (C)$$

This soln is

$$\varphi = ye^{2x} + x^2\cos y = C.$$

Remark:

Even through an eqn

$$M(x,y)dx + N(x,y)\,dy = 0$$

may not be exact sometimes it will not be difficult to find c function u, now here zero such that,

$$u(x,y)M(x,y)dx + u(x,y)N(x,y)\,dy = 0$$

is exact such a function is called an integrating factor. P. 196

2.c) $(x^2 - 2xy + 3y^2)\,dx + (y^2 + 6xy - x^2)\,dy = 0$

Soln:

$M = x^2 - 2xy + 3y^2$, $\quad N = y^2 + 6xy - x^2$

$\frac{\partial M}{\partial y} = -2x + 6y$ $\quad \frac{\partial N}{\partial x} = 6y - 2x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ equation is exact.

$\int M dx + \int N dy$ (not containing x) $= 0$

$$\Rightarrow \frac{x^3}{3} - 2y\frac{x^2}{2} + 3y^2x + \int y^2 dy = 0$$

$$\Rightarrow \frac{x^3}{3} - x^2y + 3y^2x + \frac{y^3}{3} = c$$

$y' = \frac{3x^2 - 2xy}{x^2 - 2y}$

$M_y = N_x = -2x$

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$= 3x^2 - 2xy$

$\Rightarrow F(x,y) = x^3 - x^2y + f(y)$

$\Rightarrow f'(y) = 2y$

$\Rightarrow f(y) = y^2$

2.d) Find an integrating factor for the following eqn

Solve $(2y^3+2)dx + 3xy^2dy = 0$

32

Soln:

We have $2y^3dx + 3xy^2dy + 2dx = 0$

multiplying by 'x'

$\Rightarrow 2xy^3dx + 3x^2y^2dy + 2x\,dx = 0$

$d(x^2y^3) + d(x^2) = 0$

$\Rightarrow$ Soln $x^3y^3 + x^2 = c$

P.199 Q.4 a, b.

2.e) under the same conditions as ~~show~~ that a function u on placing continuous first partial derivatives, is an if the eqn

$M(x,y)dx + N(x,y)dy = 0$ has an integrating factor u, which is a function of x alone Then $P = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is continuous fun of x alone. If P is a continuous and independent of y $\exists$ an integrating factor is given by $u(x) = e^{p(x)}$ where p is any fun satisfying $p' = P$

Soln:

Given u is integrating factor of the eqn

$M(x,y)dx + N(x,y)dy = 0$ iff

$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \rightarrow (1)$

Since u is a fun of x ~~above~~ alone

$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = u'$

(1) $\Rightarrow$ RHS $= Nu' - 0$.

Now, $P = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{N}\left[\frac{N}{u}\cdot u'\right]$

$P = \frac{1}{u}\cdot u'$ using (1)

which is clearly function of x ~~above~~ alone

$\log u = \int p\,dx$

$u = e^{\int p\,dx}$

$u = e^{p(x)}$

where $P = p'$

Rules for finding integrating factors

(i) When $Mdx + Ndy = 0$ and the eqn is homogeneous then $\frac{1}{Mx + Ny}$ is an integrating factor of $Mdx + Ndy$

(ii) When $Mdx - Ndy \neq 0$ and the eqn is of the form $f_1(x,y)\,y\,dx + f_2(x,y)\,x\,dy$ then $\frac{1}{Mx - Ny}$ is an integrating factors

(iii) If $P = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x along then $u = e^{\int P dx}$ is an integrating factor.

(iv) If $q = \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y along then $u = e^{\int q dy}$ is an integrating factors.

(v) If the eqn $Mdx + Ndy = 0$ can be rearrange in the form

$$x^a y^b (mydx + nxdy) + x^e y^e (pydx + qxdy) = 0$$

Where a, b, e, m, n, p, q are all constants then x^h, y^k is an integrating factor.

Here the constants h and k can be found using $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and equating the coefficients of like powers.

1.f) Find an integrating factor of $(e^y + xe^y)dx + xe^y dy = 0$ and hence solve it.

Soln:

Comparing with $Mdx + Ndy = 0$

$M = e^y + xe^y$, $N = xe^y$

$\frac{\partial M}{\partial y} = e^y + xe^y$, $\frac{\partial N}{\partial x} = e^y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ The eqn is not exact

Let $P = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$

$= \frac{1}{xe^y}\left[e^y + xe^y - e^y\right] = 1$

Integrating factor is $u = e^{\int P\,dx}$

$\Rightarrow u = e^{\int dx} = e^{x}$ (33)

multiplying The given differential eqn by e^{x}

$$e^{x}(e^{y} + xe^{y})dx + e^{x}\cdot xe^{y}\,dy = 0$$

$$\Rightarrow e^{x+y}dx + xe^{x+y}dx + xe^{x+y}dy = 0$$

$M = e^{x+y} + xe^{x+y}$

$\frac{\partial M}{\partial y} = e^{x+y} + xe^{x+y}$

$N = xe^{x+y}$

$\frac{\partial N}{\partial x} = e^{x+y} + xe^{x+y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is exact eqn

$\div$ by xe^{x+y}

$$\frac{1}{x}dx + dx + dy = 0$$

$$\left(\frac{1}{x} + 1\right)dx + dy = 0$$

Integrating we get,

$$\log x + x + y = \log c$$

$$x + y = \log c - \log x$$

$$x + y = \log(c/x)$$

$$e^{x+y} = c/x$$

$$c = xe^{x+y}$$

Section-4

The methods of successive approximation:

Here The consider The general form of finding soln of The eqn

$$y' = f(x, y) \rightarrow ①$$

Where f is continuous real valued function defind on some rectangle.

$R : |x - x_0| \leq a, \quad |y - y_0| \leq b \quad (a, b > 0)$ in The real (x, y) plane.

Our objective is to S.T on some interval I Containing x_0, There is a soln ϕ of ① satisfying

$$\phi(x_0) = y_0 \rightarrow ②$$

By this we mean that there is a real valued diff function ϕ satisfying ② such that the points $(x, \phi(x))$ are in R for x in I and

$$\phi'(x) = f(x, \phi(x)) \text{ for all } x \text{ in } I$$

Such a function ϕ is called a soln to the initial value problem

$$y' = (x, y) \quad , y(x_0) = y_0 \rightarrow ③ \quad \text{on } I$$

By a soln of this eqn on I, we mean that a real valued continuous function ϕ on I such that $[x, \phi(x)]$ is in R $\forall$ x in I and

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt \rightarrow ⑤ \quad \forall\ x \text{ in } I.$$

Theorem: 4

A function ϕ is a soln of the IVP $y' = f(x, y)$ $y(x_0) = y_0$ on an interval I iff it is a soln of the integral eqn.

$$y = y_0 + \int_{x_0}^{x} f(t, y)dt \quad \text{on } I$$

Proof:

Suppose ϕ is a soln of the IVP on I

Then $\phi'(c) = f(t, \phi(t)) \rightarrow ①$ on I

Since ϕ is continuous on I and f is continuous on R

The function F defined by

$F(t) = f(t, \phi(t))$ is continuous on I

Integrating ① from x_0 to x we obtain

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))dt \rightarrow ②$$

Since $\phi(x_0) = y_0$ we see that ϕ is a soln of

$$y = y_0 + \int_{x_0}^{x} f(t, y)dt \rightarrow ③$$

conversely.

suppose φ satisfies

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt \rightarrow (1) \quad \text{on } I.$$

diff using the fundamental theorem of integral calculus we find that.

$$\varphi'(x) = f(x, \varphi(x)) \quad \forall \; x \text{ in } I$$

moreover from (1), it is clear that.

$$\varphi(x_0) = y_0$$

and thus φ is a soln of the IVP

$$y' = f(x, y) \rightarrow y(x_0) = y_0 \quad \text{on } I$$

This proves the theorem.

Working rule:

let us consider solving. $y = y_0 + \int_{x_0}^{x} f(t, y)dt \rightarrow (1)$

is a I^{st} approximation to a soln we consider the fun φ_0 defined by $\varphi_0(x) = y_0$

This fun satisfies the initial condition $\varphi_0(x_0) = y_0$ but does not in general satisfy (1).

However if we compute

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_0(t))dt \quad \text{where } \varphi_0(x) = y_0$$

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$$

we might expect that φ_1 is a closes approximation is a soln than φ_0.

If we continue the process and define successively.

$$\left.\begin{aligned} &\varphi_0(x) = y \\ &\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_k(t))dt. \quad k = 0, 1, 2 \ldots \end{aligned}\right\} \rightarrow (2)$$

We might expect on taking the limits $k \to \infty$ that we would obtain $\varphi_k(x) \to \varphi(x)$

where φ would satisfy

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\,dt$$

Thus φ would be our desired soln.

We call the fun $\varphi_0, \varphi_1 \ldots$ defined by ② as successive approximation to a soln of the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t, y)\,dt$$

(or) The initial value problem $y' = f(x, y)$, $y(x_0) = y_0$.

This method is also known as Picards methods of successive approximation.

To show

There is an interval I containing x_0 where all the function $\varphi_k \Rightarrow k = 0, 1, 2 \ldots$ defined by ② exists.

Since f is continuous on R, it is bounded there

i.e) $\exists$ a constant $M > 0$. $\exists$ $|f(x,y)| \le M$, $\forall (x, y)$ in R.

Let α be the smaller of the two number a, b/m ~~between~~

Then we P.T all the φ_k are defined as $|x - x_0| \le \alpha$.

Theorem: 5

The successive approximation φ_0 defined by $y = y_0 + \int_{x_0}^{x} f(t, y)\,dt$ ①
exists as continuous function on $I = |x - x_0| \le \alpha = \min\{a, b/m\}$ and $\{x, \varphi_k(x)\}$ is in k for x in I, Indeed, the φ_k satisfy

$$|\varphi_k(x) - y_0| \le M|x - x_0| \quad \forall\ x \text{ in } I$$

① $\Rightarrow$ $\varphi_0(x) = y_0$

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_k(t))\,dt \qquad k = 0, 1, 2 \ldots$$

Proof:

clearly φ_0 exists on I as a continuous function and satisfy with $k=0$

Now, $\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$ and

$$\varphi_1(x) - y_0 = \int_{x_0}^{x} f(t, y_0)\,dt$$

Hence $$|\varphi_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0)\,dt\right|$$

$$\le \int_{x_0}^{x} |f(t, y_0)\,dt|$$

$$\le M \int_{x_0}^{x} |dt| = M|x - x_0|$$

(i) $|\varphi_1(x) - y_0| \le M|x - x_0|$

Which shows that φ_1 satisfies the inequality

since f is continuous in R the fun F_0 defined by

$F_0(t) = f(t, y_0)$ is continuous on I

Thus φ_1 which is given by

$\varphi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)\,dt$ is continuous on I.

~~Thus~~ ~~φ_1~~ Now assume the theorem has been proved for the functions $\varphi_0, \varphi_1, \dots \varphi_k$

we prove it is valid for φ_{k+1}

Indeed the proof is just a renotation of the above.

W.K.T $(t, \varphi_k(t))$ is in R for t in I

Thus the fun F_k is given by

$F_k(t) = f(t, \varphi_k(t))$ exists for t in I

It is continuous on I, since f is continuous on R and φ_k is continuous on I.

$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)\,dt$ exists as a continuous fun on I

moreover $$|\varphi_{k+1}(x) - y_0| \le \int_{x_0}^{x} |F_k(t)||dt|$$

$$\le M|x - x_0|$$

Which shows that φ_{k+1} satisfies (5)

Thus the theorem is proved by induction

3.a) consider The IVP $y' = 3y+1$, $y(x_0) = 2$ $[(or)\ y(0) = 2]$.

VQ

(a) Show That all Successive approximation $\varphi_0, \varphi_1, \dots \varphi_n$ exists $\forall$ real x.

(b) compute The 1^{st} four approximation $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ to The soln.

(c) compute The soln by actual method

(d) compute the result of (b) and c

Soln:

(a) since $f(x,y) = 3y+1$ is continuous on R. all successive approximations φ_k $(k = 0, 1, 2 \dots)$ exists real

b) The given eqn is written as

$$y = \varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt$$

i.e) $$\varphi(x) = 2 + \int_0^x (3y+1)\, dt$$

The Successive approximation are given by

$$\varphi_0(x) = y_0 = 2$$

$$\varphi_{k+1}(x) = 2 + \int_0^x (3\varphi_k(t) + 1)\, dt \qquad k = 0, 1, 2 \dots$$

Put $k=0$,

$$\varphi_1(x) = 2 + \int_0^x (3\times 2 + 1)\, dt = 2 + 7x$$

Put $k=1$,

$$\varphi_2(x) = 2 + \int_0^x [3(2+7t)+1]\, dt$$

$$= 2 + 7x + \frac{21}{2}x^2$$

$k=2$,

$$\varphi_3(x) = 2 + \int_0^x [3(2 + 7t + \tfrac{21}{2}t^2) + 1] dt$$

$$= 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3$$

$k=3$,

$$\varphi_4(x) = 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3 + \frac{63}{8}x^4$$

c) The given eqn is $y' - 3y = 1$ (36)

$$\frac{dy}{dx} - 3y = 1 \qquad P = -3,\ Q = 1 \qquad I.F = e^{\int -3dx} = e^{-3x}$$

$$y(I.F) = \int Q(I.F)dx + c$$

$$\Rightarrow y = -\frac{1}{3} + ce^{3x} \qquad ye^{-3x} = \int e^{-3x}dx + c$$

$$ye^{-3x} = \frac{-e^{-3x}}{3} + c$$

Since $x = 0,\ y = 2$

$$\boxed{y = -\frac{1}{3} + ce^{3x}}$$

$$2 = ce^{0} - \frac{1}{3}$$

$$2 + \frac{1}{3} = c$$

$$c = 7/3$$

$$\therefore\ y = -\frac{1}{3} + \frac{7}{3}e^{3x}$$

$$= \frac{1}{3}(7e^{3x} - 1)$$

$$= \frac{1}{3}\left[7\left(1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \cdots\right) - 1\right]$$

$$= \frac{1}{3}\left[7 + 21x + \frac{63x^2}{2} + \frac{189x^3}{6} + \cdots - 1\right] \qquad \frac{7}{3} - \frac{1}{3} = \frac{6}{3} = 2$$

$$= \frac{7}{3} + 7x + \frac{21}{2}x^2 + \frac{21x^3}{2} + \frac{63x^4}{8} + \cdots - \frac{1}{3}$$

$$y = 2 + 7x + \frac{21}{2}x^2 + \frac{21x^3}{2} + \cdots$$

Also from (b)

$$\varphi(x) = 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3 + \cdots$$

$$= \frac{1}{3}[7e^{3x} - 1]$$

3.b) Find the I^{st} four successive approximation $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ of the eqn. $y' = 1 + xy$, (i) $y(0) = 0$. (ii) $y(0) = 1$.

Soln:

Given $y' = 1 + xy$, $y(0) = 0$

ie) $f(x, y) = xy + 1$, $y(0) = 0$, $x_0 = 0$

The successive approximation are

$$\varphi_0(x) = 0$$

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(1 + t\varphi_k(x))\,dt$$

$k = 0$, $\quad \varphi_1(x) = 0 + \int_0^x f(1 + t\varphi_0(t))\,dt$

c) The given eqn is $y' - 3y = 1$ (36)

$$\frac{dy}{dx} - 3y = 1$$

$P = -3, Q = 1$ $I.F = e^{\int -3dx} = e^{-3x}$

$y(I.F) = \int Q(I.F)dx + c$

$y e^{-3x} = \int e^{-3x} dx + c$

$y e^{-3x} = \frac{-e^{-3x}}{3} + c$

$\boxed{y = -\frac{1}{3} + ce^{3x}}$

$$\Rightarrow y = -\frac{1}{3} + ce^{3x}$$

Since $x = 0,\ y = 2$

$$2 = ce^{0} - \frac{1}{3}$$

$$2 + \frac{1}{3} = c$$

$$c = 7/3$$

$$\therefore y = -\frac{1}{3} + \frac{7}{3}e^{3x}$$

$$= \frac{1}{3}(7e^{3x} - 1)$$

$$= \frac{1}{3}\left[7\left(1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \dots\right) - 1\right]$$

$$= \frac{1}{3}\left[7 + 21x + \frac{63x^2}{2} + \frac{189x^3}{6} + \dots - 1\right]$$

$\frac{7}{3} - \frac{1}{3} \Rightarrow \frac{6}{3} = 2$

$$= \frac{7}{3} + 7x + \frac{21}{2}x^2 + \frac{21x^3}{2} + \frac{63x^4}{8} + \dots - \frac{1}{3}$$

$$y = 2 + 7x + \frac{21}{2}x^2 + \frac{21x^3}{2} + \dots$$

Also from (b)

$$\varphi(x) = 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3 + \dots$$

$$= \frac{1}{3}[7e^{3x} - 1]$$

3.b) Find The I^{st} four successive approximation $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ of The eqn. $y' = 1 + xy$, (i) $y(0) = 0$. (ii) $y(0) = 1$.

Soln:

Given $y' = 1 + xy$, $y(0) = 0$

(i) $f(x, y) = xy + 1$, $y(0) = 0$, $x_0 = 0$

The successive approximation are

$$\varphi_0(x) = 0$$

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(1 + t\varphi_k(x))\, dt$$

$k = 0$, $$\varphi_1(x) = 0 + \int_0^x f(1 + t\varphi_0(t))\, dt$$

$$= \int_0^x (1 + t\,\varphi_0(t))dt$$

$$= \int_0^x (1 + t(0))dt = \int_0^x dt$$

$$\varphi_1(x) = x$$

$k = 1,$

$$\varphi_2(x) = \int_0^x (1 + t \cdot t)dt$$

$$= \int_0^x (1 + t^2)dt = x + \frac{x^3}{3}$$

$k = 2,$

$$\varphi_3(x) = \int_0^x (1 + t \cdot \varphi_2(t))dt$$

$$= \int_0^x [1 + t(t + \frac{t^3}{3})]dt = \int_0^x (1 + t^2 + \frac{t^4}{3})dt$$

$$\varphi_3(x) = x + \frac{x^3}{3} + \frac{x^5}{15}$$

Hence, $\varphi_0 = 0$, $\varphi_1 = x$, $\varphi_2 = x + \frac{x^3}{3}$, $\varphi_3 = x + \frac{x^3}{3} + \frac{x^5}{5}$

3.c) Given $y' = x + y$, $y(0) = 1$, Find by Picards method the I^{st} four successive approximation.

Soln:

Given $f(x, y) = x + y$, $y(0) = 1$, $x_0 = 0$. The Succesive approximation are.

$$\varphi_0(x) = 1$$

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^x f(t, \varphi_k(t))dt$$

$$\varphi_{k+1}(x) = 1 + \int_{x_0}^x \cancel{f}(t + \varphi_k(t))dt$$

$k = 0,$

$$\varphi_1(x) = 1 + \int_0^x (t + \varphi_0(t))dt$$

$$= 1 + \int_0^x (t + 1)dt$$

$$\varphi_1(x) = 1 + x + \frac{x^2}{2}$$

$$\varphi_2(x) = 1 + \int_0^x (t + \varphi_1(t))dt$$

$$= 1 + \int_0^x (t + 1 + t + \frac{t^2}{2})dt$$

37

$$\varphi_2(x) = 1+x+x^2+\frac{x^3}{6}$$

$$\varphi_3(x) = 1+\int_0^x (t+\varphi_2(t))\,dt$$

$$= 1+\int_0^x (t+1+t+t^2+\frac{t^3}{6})\,dt$$

$$= 1+\left[\frac{t^2}{2}+t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{24}\right]_0^x$$

$$\varphi_3(x) = 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{24}$$

3d) $y' = y^2$, $y(0)=0$

Soln:

$$\varphi_0(x) = y_0 = 0$$

$$\varphi_{k+1}(x) = y_0+\int_0^x f(t,\varphi_k(t))\,dt$$

$$\varphi_{k+1}(x) = \int_0^x \varphi_k^2(t)\,dt$$

$k=0$, $$\varphi_1(x) = \int_0^x \varphi_0^2(t)\,dt = \int_0^x 0.dt$$

$$\varphi_1(x) = 0$$

$k=1$, $$\varphi_2(x) = \int_0^x \varphi_1^2(t)\,dt = \int_0^x 0.dt$$

$$\varphi_2(x) = 0.$$

Similarly, $\varphi_3 = 0$

$$\therefore \varphi_1 = \varphi_2 = \varphi_3 = 0.$$

3e) $y' = y^2$, $y(0)=1$, $y_0 = \varphi_0(t) = 1$, $x_0 = 0$

Soln:

$$\varphi_{k+1}(x) = y_0+\int_{x_0}^x f(t,\varphi_k(t))\,dt$$

$k=0 \Rightarrow$ $$\varphi_1(x) = 1+\int_0^x f(t,\varphi_0(t))\,dt \Rightarrow 1+\int_0^x 1.dt$$

$$\varphi_1(x) = 1+[t]_0^x = 1+x$$

$k=1$, $$\varphi_2(x) = 1+\int_0^x f(t,\varphi_1(t))\,dt \Rightarrow 1+\int_0^x [1+t]^2 dt = 1+\int_0^x (t^2+2t+1)\,dt$$

$$= 1+\left[\frac{t^3}{3}+\frac{2t^2}{2}+t\right]_0^x = 1+t+t^2+\frac{t^3}{3}$$

Section-5

The Lipschitz Condition

Definition:

Let f be a function defined for (x,y) in a set S we say that f satisfies a respectively condition on S if

$\exists$ a constant $K > 0 \ni$

The constant K is called a Lipschitz constant, the condition is Lipschitz condition.

Theorem: 6

Suppose S is either a rectangle $|x-x_0| \le a$, $|y-y_0| \le b$ $(a,b>0)$ or a strip $|x-x_0| \le a$, $|y| \le \infty$ $(a>0)$ and so that for a real valued function defined on S $\exists$ $\frac{\partial f}{\partial y}$ exists is continuous on S and

$\left|\frac{\partial f}{\partial y}(x,y)\right| \le K$, $(x,y) \in S$ for some $K>0$ Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

Soln:

We have,

$$f(x,y_1) - f(x,y_2) = \int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x,t)\,dt$$

and hence

$$|f(x,y_1) - f(x,y_2)| \le \int_{y_2}^{y_1} \left|\frac{\partial f}{\partial y}(x,t)\right| |dt|$$

$$\le K|y_1 - y_2| \quad \forall (x,y_1)(x,y_2) \text{ in } S$$

Hence f satisfies the Lipschitz condition.

Problems

4.a) If S is defined by the rectangle $|x| \leq a$, $|y| \leq b$ S.T the function $f(x,y) = x^2+y^2$ satisfies lipschitz condition. Find the lipschitz constant

Soln:

Given $f(x,y) = x^2+y^2$, $|x| \leq a$, $|y| \leq b$

$\Rightarrow \frac{\partial f}{\partial y}(x,y) = 2y$

$\therefore \left|\frac{\partial f}{\partial y}(x,y)\right| = |2y| \leq 2b$ for $(x,y) \in S$

Hence $f(x,y)$ satisfies the lipschitz condition with lipschitz constant $K = 2b$

4.b) S.T $f(x,y) = xy^2$ satisfies lipschitz condition on the rectangle $|x| \leq 1$, $|y| \leq 1$ but does not satisfy a lipschitz condition on the strip $|x| \leq 1$, $y < \infty$

Soln:

$f(x,y) = xy^2$

$S: |x| \leq 1$, $|y| \leq 1$

$\Rightarrow \frac{\partial f}{\partial y}(x,y) = 2xy$

Hence $\left|\frac{\partial f}{\partial y}(x,y)\right| = |2xy| \leq 2$ for $(x,y) \in S$

$\therefore$ $f(x,y)$ satisfies lipschitz condition wise $K=2$ if S is the strip $|x| \leq 1$, $y < \infty$

We have $\left|\frac{f(x,y_1) - f(x,0)}{f(y_1 - 0)}\right| = |x||y|$ which $\to \infty$ as $|y| \to \infty$ if $|x| \neq 0$

Hence the lipschitz condition is not satisfied on the strip $|x| \leq 1$, $|y| < \infty$.

4.c) S.T The function $f(x,y) = y^{2/3}$ even though continuous on $R: |x| \leq 1, |y| \leq 1$ does not satisfy lipschitz condition there

Soln:

Let $y_1 > 0$

$$\left|\frac{f(x,y_1) - f(x,0)}{(y_1 - 0)}\right| = \frac{y_1^{2/3}}{y_1} = \frac{1}{y_1^{1/3}} = \frac{1}{0} = \infty$$

which is unbounded as $y_1 \to 0$

S.T function f given by $f(x,y) = y^{1/2}$ does not satisfy the lipschitz condition on $R: |x| \leq 1, 0 \leq y \leq 1$

(ii) Show this f satisfies lipschitz condition on any rectangle R of the form

$R: |x| \leq a, \quad b \leq y \leq c, \quad a, b, c > 0$

4.d) S.T the function f given by $f(x,y) = x^2|y|$ satisfies a lipschitz condition on $R: |x| \leq 1, |y| \leq 1$ even though $\frac{\partial f}{\partial y}$ does not exist at $(x,0)$ if $x \neq 0$.

Soln:

$$|f(x,y_1) - f(x,y_2)| = |x^2|y_1| - x^2|y_2||$$

$$= |x^2|\,||y_1| - |y_2||$$

$$\leq |x^2|\,|y_1 - y_2|$$

$$\leq |y_1 - y_2| \quad \text{since } |x| \leq 1$$

Thus f satisfies the lipschitz condition with $k = 1$

Now for $x \neq 0$.

$$\left(\frac{\partial f}{\partial y}\right)_{(x,0)} = \lim_{y \to 0} \frac{f(x,y) - f(x,0)}{y - 0}$$

$$= \frac{x^2|y|}{y} = \pm x^2$$

$\therefore \frac{\partial f}{\partial y}$ does not exist at $(x,0)$

This Problem S.T. the condition $\left(\frac{\partial f}{\partial y}\right) \le K$ in Thm 6 is only sufficient but not necessary for the validity of Lipschitz condition (29)

---

20) Under the conditions that a function
(i) u on R having continuous first partial derivatives, is an integrating factor if and only if

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \text{ on } R.$$

S.T. if the equation $M(\ )dx + N(\ )dy = 0$ has an I.F. u which is a function of x alone then $p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a continuous function of x alone.

(ii) If p is continuous and depends only on x, S.T. an I.F. is given by $u(x) = e^{P(x)}$ where P is any function satisfying $P' = p$.

HW: 5 a) b).